LANGUAGE AND UNDERSTANDING OF MATHEMATICAL CONCEPTS

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One may argue that mathematics has two sides, concepts and content, which cannot be separated from each other. In this paper I will focus on the role of the language in the process of understanding mathematical concepts.

Mathematical understanding

No matter whom you ask, when it comes to mathematics education "understanding" seems to be the main issue. Students like mathematics as long as they "understand" and understanding is also one major goal of mathematics teaching at all levels in the educational system (Stadler, 2002). A common opinion among peers and students is that mathematics consists of individual work devoted to solving tasks with only one correct answer (Schoenfeld, 1992). In this case mathematical understanding equals skills to use algorithms and getting the right answer. Educators and researchers in mathematics tend to seek a deeper understanding of the content associated with a certain concept (Sfard, 1994; Stadler, 2002). This dualistic view of mathematical understanding can be described in terms of instrumental and relational understanding (Skemp 1997, Pesek & Kirshner, 2000).

One way to capture mathematical understanding is to describe it as a process where a mathematical object transforms from being a process to become a mental object (Sfard, 1991; Tall et al, 2000). Thus, a deep understanding of a mathematical object is not mainly about manipulating complicated expressions. Instead, to be able to create an internal picture of an abstract mathematical concept seems to be a major part of mathematical ability. Sfard (1991) describes the development from an operational to a structural conception as a process of reification. In the stage of interiorization, the learner gets acquainted with the concept by computing in single steps. These steps get associated with each other in the next condensation phase. Reification is an ability to see the concept as a whole. It is a static state where 'the concept becomes semantically unified by this abstract and purely imaginary construct' (Sfard, 1991, p 20).

Language and understanding

An important part of the mathematical language is all its signs and symbols. A question, which has occupied many philosophers of mathematics, is what comes first – the mathematical symbol or the meaning of the symbol? From an objectivistic point of view learning and understanding are to build links between symbols and a reality, which exists independently of our minds. Another possible option is to look the other way around. It is our understanding, based upon experience, which fills signs and notions with their particular meaning (Lakoff & Johnson, 1980).

Mathematical discourse and its objects are mutually constituted (Sfard, 1998). It is the discursive activity that creates the need for mathematical objects and it is the mathematical objects that influence the mathematical discourse and leads it in new directions. To name mathematical objects is more of a conceptual process than a question of baptism. When new concepts are introduced, the learner tries to relate them to familiar templates to use in the new discourse. Hence, the introduction of mathematical symbols can be considered as an important part of the reification process.

Mathematical understanding is both a linguistic and a conceptual matter (Vergnaud, 1998). To understand mathematics, a learner must be able to identify relationships between the mathematical symbols, but also their relation to natural everyday language. There is a strong relationship between mathematical knowledge and language. A lot of students' mathematical knowledge is of an implicit kind. However, to be able to discuss and argue about mathematics demands an ability to put mathematical knowledge into words and symbols. The status of mathematical knowledge changes as it becomes more explicit.

Núñez et al (1999) states that 'embodiment provides a deep understanding of what human ideas are, and how they are organized in vast (most unconscious) conceptual systems grounded in physical, lived reality' (p 50). To grasp understanding of abstract concepts, for example within mathematics, metaphors play an important role (Lakoff & Johnson, 1980). A metaphor expresses a given idea in terms of something already familiar to the learner. It serves as a bridge between concrete experience and abstract understanding.

The relation between metaphors and reification has also been stressed by Sfard (1994), who states that 'the metaphor of an ontological object is indispensable for the kind of understanding people are prepared to call "deep" or "true"' (p 54). She continues 'Reification is, in fact, the birth of a metaphor which brings a mathematical object into existence and thereby deepens our understanding' (Sfard 1994, p 54)

A study of students' understanding of cosine and the unit circle
The following examples are taken from a previous study about students understanding of trigonometry and the unit circle (Stadler, 2002)[1]. Groups with three students in each were given three drawn triangles (See figure 1-3). Their task was to determine the cosine value for the marked angels. The triangles were drawn on half centimetre squared paper; rulers and protractors were placed in front of the students on the table, but they were not allowed to use calculators or any formulae.

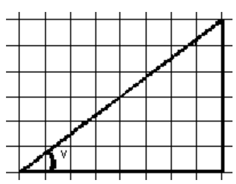

Figure 1

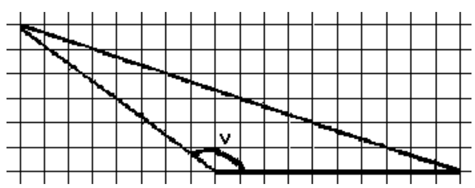

Figure 2

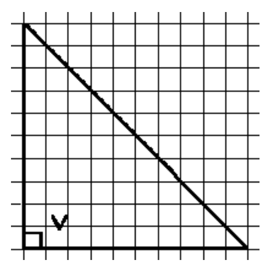

Figure 3

The transcriptions were reanalysed with the aim to study students use of language and their understanding of mathematical concepts interact. Three questions of interest are:

1. How do the students understand the concept of cosine?
2. How does the switch between implicit and explicit language affect the students understanding?
3. How do the students use metaphors to understand a mathematical concept?

[1] The aim of the study was to investigate students' understanding of trigonometry and the unit circle. The analysis was made from partly another perspective. By means of intentional analysis, I analysed the students' conceptual, situated and cultural context in which they solved the tasks.

To ascribe meaning to the students' actions, I have used "Intentional analysis". The theoretical framework for an intentional perspective can be found in von Wrights work (1971). von Wright emphasizes that we are not seeking for a cause-relation, where an event causes an other event. Instead, we ask what the person intends to achieve with what she does and says. It is the person's intentions with her acts or statements that is the object of interest (see Halldén, Scheja & Jakobsson Öhrn, 2001).

The acute angle triangle

Three boys, Daniel, Niklas and Tomas, were trying to find out the cosine value for an acute angle in a right angle triangle. The triangle legs were three and four centimetres and the hypotenuse was five centimetres (see fig 1).

Daniel determined the length of the sides in the triangle by counting the squares. On a paper he wrote that the angle was $v = \sin^{-1}\left(\frac{8}{6}\right)$ and concluded that $\cos v = \cos\left(\sin^{-1}\left(\frac{8}{6}\right)\right)$. But this answer did not please Niklas.

Niklas: *Let's see… That was the cosine value. Positive. But you don't need that. You don't need sine powers raised to one.*
Daniel: (mutters).
Niklas: *Cosine, that is adjacent divided with the hypotenuse…*
Daniel: *Yes*.
Niklas: … *and that is 6. So erase sine.*

The group continued with their work. After a while they returned to the first task and discovered the wrong quotient.

Niklas: *Cosine is adjacent divided with the hypotenuse*.
Daniel: *It is cos four through five*.
Niklas: *Yes, zero eight*.
Tomas: *And then you put cos in front of everything.*
The group wrote the answer "*cos 0,8*".
I claimed that their answer could be interpreted as they had counted the cosine value for the angle 0,8°.
Tomas: *Shall we write cos v equals…*
Daniel: *But it is equal to v*.
Niklas: *Mm…*
Tomas: *Yes, but it isn't the angle. It isn't rased to of degrees. In that case I would have…*
Niklas: *Yes, now! Erase the cosine. The cosine-value is 0,8.*

As much as mathematics is about its objects it is just as much about the relationship between the objects. In the case of cosine, the students need the concepts of angle, cosine for an angle (or an arbitrary number) and the inverse cosine. They also need to know how the relationships between these concepts are organized. Ignoring the wrong quotient, the first answer could be regarded as correct. With a genuine understanding of the trigonometric functions, the boys would have realised that the complicated expression they produced could be replaced by the quotient of the adjacent and the hypotenuse. Thus, the quotient $8/10$ would immediately have given them the cosine value. The lack of understanding is obvious when they "erase sine". The expression left is devoid of meaning.

The attentive reader may have noticed that the boys use the spoken expression “cosine is adjacent divided with the hypotenuse”. From a conceptual perspective this causes trouble. It is no longer clear whether cosine value can be computed in this way or if it is the cosine value. When the boys answer “cos 0,8” they had actually just answered according to their own definition. One explanation of the above statement is that it expresses the difference between informal spoken and the more formal written language. Our oral mathematical discussions tend to be less exact and not always formally correct.

One interpretation of why Tomas wants to put “*cos in front of everything*” is that he regards cosine as a label for what they have computed. The answer 0.8 seems too meaningless, without content. To put cosine in front of 0.8 is his attempt to tell what has been done. He seems unaware of the conceptual change this makes. It is possible that the others have the same conceptual understanding of cosine as Tomas. To find the cosine value is to write “*cos*” and the value. Thus, they use cos as a label for what has been calculated.

Dealing with sides and angles in a right-angled triangle can be regarded as standard tasks within trigonometry. In fact, these kinds of tasks often introduce the subject area of trigonometry, although they are normally designed in a slightly different way. Students practise the formula cos v = adjacent leg / hypotenuse, in a mechanical way. The boys do not seem to have realised that cos v is the simplest thing to count, because they have not been forced to think of what the expression cos v really represent.

The right angle triangle

Two girls and one boy, Jenny, Anna and Björn, were working with the right-angled triangle (see fig 2). They seemed relieved when they saw the figure, implying that they all recognized cosine for $v = 90^{\circ}$.
Jenny: *Cosine 90 is zero, isn't it?*
Björn: *Yes, cos zero is 90.*
Jenny: *No, it is the other way around. Cos 90 degrees is zero.*

Björn said that he could prove his argument by drawing a unit circle, but it became Jenny who showed her conclusion by marking the angle in the circle. She explained to the others in the group how to find out the cosine value for an angle in the unit circle.
Jenny: *It works like this... if you want cos. If you have a unit circle, then you get cosine on this axis (points on the x-axis). And here you can see that cos 90 is zero. Imagine that this is an arrow that turns around. And it turns up here, and you get the cosine-value on the x-axis.*
Anna: *But there it is zero. It can be zero for many angles. Cos 90 is zero.*
Jenny: *Yes.*
Björn: *Listen to me, Cos zero is 90!*

At a first glance it seems as if Björn's understanding of cosine and the cosine value is confused. The lack of any relational understanding appears obvious. Neither does he seem to have an instrumental understanding as he fails to “put the right number in the right places in the expression”. However, Björn's statement may be a linguistic mistake instead of conceptual one. It is possible that Björn imagines the question “For which value of the angle is cosine zero?” and then answers “Well, cosine is zero when the angle is 90”. Things are being written in the order they are being mentioned. The result is that Bjorn's answers cos zero is 90.

Can the unit circle be regarded as a metaphorical construction within trigonometry? If the answer is “yes” Jenny gives an example of how a metaphor can be used to explain a mathematical concept, which has been reified. To be able to distinguish between the angles from the cosine value in an expression is a very abstract thing to do. The unit circle offers a concrete model, where it is possible to point to the angle and the cosine value. Though the unit circle has been an educational subject to all the students, it does not seem to support Björn’s and Anna’s understanding of the cosine value. Most angles can be clearly viewed in it, but for the angles $n \cdot 90^\circ$ the picture can be confusing, because the angles coincide with the axis. In some way, the angles disappear and everything seems to be zero.

Discussion

An operational conception of the concepts of cosine means that the students can use it for computations, sometimes with the help of detailed step-by-step instructions (Sfard, 1991). In the absence of a deeper understanding it is difficult to use words, signs and symbols in an appropriate way. Instead they use the most familiar template available at the moment, which may gives rise to an incorrect representation and use of language. Vergnaud (1998) emphasizes the relationship between mathematical knowledge and language. In oral discussions, students tend to use a more informal language where important words are left out. When the spoken words are put into written language, mistakes may follow and lead to wrong conclusions.

In the next future I will continue to build my theoretical framework about language, metaphors and reification. However, Lakoff and Johnson (1980) stress that metaphors play an important role in grasping understanding of abstract concepts. Sfard (1994) states that reifications can be regarded as the birth of metaphor. Are we using different metaphors for gaining understanding and others to objectify the understanding we have reached?

The examples discussed above should not be misunderstood as any kinds of attempts to generalize students’ understanding of cosine and the unit circle. The intention of presenting these examples was to demonstrate the relationship between language and understanding of mathematical concepts. The non-striking results may be a consequence of that the empirical material actually was produced for another study. To penetrate how mathematical content and concepts interact is also my overall aim with my further research. In particular I am interested in which role the language plays in the reification process. An important question for my future research is how I shall design a study to investigate this relationship.

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